CAMBRIDGE ASSESSMENT

STEP Solutions 2010

Mathematics
STEP 9465/9470/9475

October 2010

TEST

Hints & Solutions for STEP II 2010

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1 When two curves meet they share common coordinates; when they "touch" they also share a common gradient. In the case of the *osculating circle*, they also have a common curvature at the point of contact. Since curvature (a further maths topic) is a function of both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, the question merely states that C and its osculating circle at P have equal rates of change of gradient. It makes sense then to differentiate twice both the equation for C and that for a circle, with equation of the form $(x-a)^2 + (y-b)^2 = r^2$, and then equate them when $x = \frac{1}{4}\pi$. The three resulting equations in the three unknowns a, b and r then simply need to be solved simultaneously.

For $y = 1 - x + \tan x$, $\frac{dy}{dx} = -1 + \sec^2 x$ and $\frac{d^2y}{dx^2} = 2\sec^2 x \tan x$.

For $(x-a)^2 + (y-b)^2 = r^2$, $2(x-a) + 2(y-b)\frac{dy}{dx} = 0$ and $2 + 2(y-b)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$.

When $x = \frac{1}{4}\pi$, $y = 2 - \frac{1}{4}\pi$ and so $\left(\frac{1}{4}\pi - a\right)^2 + \left(2 - \frac{1}{4}\pi - b\right)^2 = r^2$;

$$\frac{dy}{dx} = -\frac{(x-a)}{(y-b)} = 1 \text{ then gives a relationship between } a \text{ and } b;$$

and $\frac{d^2y}{dx^2} = 4 = -\frac{4}{2(y-b)}$ gives the value of b.

Working back then gives a and r.

Answers: The osculating circle to C at P has centre $\left(\frac{1}{4}\pi - \frac{1}{2},\ \frac{5}{2} - \frac{1}{4}\pi\right)$ and radius $\frac{1}{\sqrt{2}}$.

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2 The single-maths approach to the very first part is to use the standard trig. "Addition" formulae for sine and cosine, and then to use these results, twice, in (i); firstly, to rewrite $\sin^3 x$ in terms of $\sin 3x$ so that direct integration can be undertaken; then to express $\cos 3x$ in terms of $\cos^3 x$ in order to get the required "polynomial" in $\cos x$. Using the given "misunderstanding" in (ii) then leads to a second such polynomial which, when equated to the first, gives an equation for which a couple of roots have already been flagged. Unfortunately, the several versions of the question that were tried, in order to help candidates, ultimately led to the inadvertent disappearance of the interval 0 to π in which answers had originally been intended. This meant that there was a little bit more work to be done at the end than was initially planned.

$\cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x = (2c^2 - 1)c - 2sc.s = (2c^2 - 1)c - 2c(1 - c^2)$
$= 4c^3 - 3c$.

$\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x = 2sc.c + (1 - 2s^2)s = 2s(1 - s^2) + s(1 - 2s^2)$
$= 3s - 4s^3$

(i) $I(\alpha) = \int_0^\alpha \left(7\sin x - 8\sin^3 x\right) dx = \int_0^\alpha \left(\sin x + 2\sin 3x\right) dx = \left[-\cos x - \frac{2}{3}\cos 3x\right]_0^\alpha$

$= -\cos\alpha - \frac{2}{3}(4\cos^3\alpha - 3\cos\alpha) + 1 + \frac{2}{3} = -\frac{8}{3}c^3 + c + \frac{5}{3}$

and $I(\alpha) = 0$ when $c = 1$ $(\alpha = 0)$

(ii) $J(\alpha) = \left[\tfrac{7}{2}\sin^2 x - \tfrac{8}{4}\sin^4 x\right]_0^{\alpha} = \tfrac{7}{2}(1 - \cos^2\alpha) - 2(1 - \cos^2\alpha)^2 = -2c^4 + \tfrac{1}{2}c^2 + \tfrac{3}{2}$

$I(\alpha) = J(\alpha) \Rightarrow 0 = 12c^4 - 16c^3 - 3c^2 + 6c + 1 = (c-1)^2(2c+1)(6c+1)$

Thus $\cos\alpha = 1$, $\alpha = 0$; $\cos\alpha = -\tfrac{1}{2}$, $\alpha = \tfrac{2}{3}\pi$; and $\cos\alpha = -\tfrac{1}{6}$, $\alpha = \pi - \cos^{-1}(\tfrac{1}{6})$.

Answers: $\alpha = 2n\pi,\ 2n\pi \pm \tfrac{2}{3}\pi,\ (2n+1)\pi \pm \cos^{-1}\tfrac{1}{6}$

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3 You don't have to have too wide an experience of mathematics to be able to recognise the *Fibonacci Numbers* in a modest disguise here. (However, this is of little help here, as you should be looking to follow the guidance of the question.) In (i), you are clearly intended to begin by substituting n = 0, 1, 2 and 3, in turn, into the given formula for F_n, using the four given terms of the sequence. You now have four equations in four unknowns, and the given result in (i) is intended to help you make progress; with (ii) having you check the formula in a further case. In the final part, you should split the summation into two parts, each of which is an infinite geometric progression.

(i) $F_0 = 0 \Rightarrow 0 = a + b$ or $b = -a$. Then $F_1 = 1 \Rightarrow 1 = a(\lambda - \mu)$.
$[F_2 = 1 \Rightarrow 1 = a(\lambda^2 - \mu^2) \Rightarrow \lambda + \mu = 1$ is needed later$]$

and $F_3 = 2 \Rightarrow 2 = a(\lambda^3 - \mu^3) = a(\lambda - \mu)(\lambda^2 + \lambda\mu + \mu^2)$ by the difference of two cubes

$= 1.(\lambda^2 + \lambda\mu + \mu^2) \Rightarrow \lambda^2 + \lambda\mu + \mu^2 = 2$

Then, using any two suitable eqns., e.g. any two of $\lambda\mu = -1$, $\lambda - \mu = \dfrac{1}{a}$ and $\lambda + \mu = 1$, and solving simultaneously gives $a = \dfrac{1}{\sqrt{5}}$, $b = -\dfrac{1}{\sqrt{5}}$, $\lambda = \dfrac{1}{2}\left(1 + \sqrt{5}\right)$, $\mu = \dfrac{1}{2}\left(1 - \sqrt{5}\right)$.

(ii) Using the formula $F_n = a\lambda^n + b\mu^n = \dfrac{1}{2^n\sqrt{5}}\left\{(1+\sqrt{5})^n - (1-\sqrt{5})^n\right\}$ with $n = 6$; the *Binomial Theorem* gives $\left(1+\sqrt{5}\right)^6 = 1 + 6\sqrt{5} + 15.5 + 20.5\sqrt{5} + 15.5^2 + 6.5^2\sqrt{5} + 5^3 = 576 + 256\sqrt{5}$.

Similarly, $\left(1-\sqrt{5}\right)^6 = 576 - 256\sqrt{5}$ so that $F_6 = \dfrac{1}{2^6\sqrt{5}}\left\{512\sqrt{5}\right\} = 8$.

(iii) $\displaystyle\sum_{n=0}^{\infty}\frac{F_n}{2^{n+1}} = \frac{a}{2}\sum_{n=0}^{\infty}\left(\frac{\lambda}{2}\right)^n - \frac{a}{2}\sum_{n=0}^{\infty}\left(\frac{\mu}{2}\right)^n = \frac{1}{2\sqrt{5}}\left(\frac{1}{1-\frac{1}{4}\left(1+\sqrt{5}\right)}\right) - \frac{1}{2\sqrt{5}}\left(\frac{1}{1-\frac{1}{4}\left(1-\sqrt{5}\right)}\right)$ using the S_∞ formula for the two GPs;

$$= \frac{1}{2\sqrt{5}}\left(\frac{4}{3-\sqrt{5}}\right) - \frac{1}{2\sqrt{5}}\left(\frac{4}{3+\sqrt{5}}\right).$$

Rationalising denominators then yields $\dfrac{2}{\sqrt{5}}\left(\dfrac{3+\sqrt{5}}{9-5}\right) - \dfrac{2}{\sqrt{5}}\left(\dfrac{3-\sqrt{5}}{9-5}\right) = \dfrac{2}{\sqrt{5}}\left(\dfrac{2\sqrt{5}}{4}\right) = 1$.

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4 Hopefully, the obvious choice is $y = a - x$ for the initial substitution and, as with any given result, you should make every effort to be clear in your working to establish it. Thereafter, the two integrals that follow in (i) use this result with differing functions and for different choices of the upper limit a. Since this may be thought an obvious way to proceed, it is (again) important that your working is clear in identifying the roles of f(x) and f($a - x$) in each case. In part (ii), however, it is not the first *result* that is to be used, but rather the *process* that yielded it. The required substitution should, again, be obvious, and then you should be trying to mimic the first process in this second situation.

(i) Using the substn. $y = a - x$, $\mathrm{d}y = -\,\mathrm{d}x$ and $(0, a) \to (a, 0)$ so that

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,\mathrm{d}x = \int_a^0 \frac{f(a-y)}{f(a-y)+f(y)}\,.-\mathrm{d}y = \int_0^a \frac{f(a-y)}{f(a-y)+f(y)}\,\mathrm{d}y$$

$= \displaystyle\int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,\mathrm{d}x$, since the x/y interchange here is nothing more than a re-labelling.

Then $2I = \displaystyle\int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,\mathrm{d}x = \int_0^a 1.\,\mathrm{d}x = [x]_0^a = a \Rightarrow I = \tfrac{1}{2}a.$

For f(x) = ln(1 + x) , $\ln(2 + x - x^2) = \ln[(1 + x)(2 - x)] = \ln(1 + x) + \ln(2 - x)$

and $\ln(2 - x) = \ln(1 + [1 - x]) = \mathrm{f}(a - x)$ with $a = 1$ so that $\displaystyle\int_0^1 \frac{f(x)}{f(x)+f(1-x)}\,\mathrm{d}x = \tfrac{1}{2}$.

$$\int_0^{\pi/2} \frac{\sin x}{\sin\left(x+\frac{1}{4}\pi\right)}\,\mathrm{d}x = \int_0^{\pi/2} \frac{\sin x}{\sin x.\frac{1}{\sqrt{2}}+\cos x.\frac{1}{\sqrt{2}}}\,\mathrm{d}x = \sqrt{2}\int_0^{\pi/2} \frac{\sin x}{\sin x+\sin\left(\frac{1}{2}\pi-x\right)}\,\mathrm{d}x = \tfrac{1}{4}\pi\sqrt{2}\,.$$

(ii) For $u = \dfrac{1}{x}$, $\mathrm{d}u = -\dfrac{1}{x^2}\mathrm{d}x$ and $\left(\tfrac{1}{2}, 2\right) \to \left(2, \tfrac{1}{2}\right)$.

Then $\displaystyle\int_{0.5}^2 \frac{1}{x}.\frac{\sin x}{\left(\sin x+\sin\left(\frac{1}{x}\right)\right)}\,\mathrm{d}x = \int_{0.5}^2 \frac{1}{x^2}.\frac{x\sin x}{\left(\sin x+\sin\left(\frac{1}{x}\right)\right)}\,\mathrm{d}x = \int_2^{0.5} \frac{\frac{1}{u}.\sin\left(\frac{1}{u}\right)}{\left(\sin\left(\frac{1}{u}\right)+\sin u\right)}\,.-\mathrm{d}u$

$= \displaystyle\int_{0.5}^2 \frac{1}{u}.\frac{\sin\left(\frac{1}{u}\right)}{\left(\sin u+\sin\left(\frac{1}{u}\right)\right)}\,\mathrm{d}u$ or $\displaystyle\int_{0.5}^2 \frac{1}{x}.\frac{\sin\left(\frac{1}{x}\right)}{\left(\sin x+\sin\left(\frac{1}{x}\right)\right)}\,\mathrm{d}x$

Adding then gives $2I = \displaystyle\int_{0.5}^2 \frac{1}{x}\,\mathrm{d}x = [\ln x]_{0.5}^{2} = 2\ln 2 \Rightarrow I = \ln 2$.

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5 The opener here is a standard bit of A-level maths using the scalar product, and the following parts use this method, but with a bit of additional imagination needed. In 3-dimensions, there are infinitely lines inclined at a given angle to another, specified line, and this is the key idea of the final part of the question. Leading up to that, in (i), you need only realise that a line equally inclined to *two* specified (non-skew) lines must lie in the plane that bisects them (and is perpendicular to the plane that contains, in this case, the points O, A and B). One might argue that the vector treatment of "planes" is further maths work, but these ideas are simple geometric ones.

$$\cos 2\alpha = \frac{(1,\ 1,\ 1)\bullet(5,\ -1,\ -1)}{\sqrt{3}.\sqrt{27}} = \frac{1}{3}$$

(i) l_1 equally inclined to OA and OB iff $\frac{(m,\ n,\ p)\bullet(1,\ 1,\ 1)}{\sqrt{m^2+n^2+p^2}.\sqrt{3}} = \frac{(m,\ n,\ p)\bullet(5,\ -1,\ -1)}{\sqrt{m^2+n^2+p^2}.\sqrt{27}}$

i.e. $3(m + n + p) = 5m - n - p$ or $m = 2(n + p)$.

For l_1 to be the angle bisector, we also require (e.g.) $\frac{m+n+p}{\sqrt{m^2+n^2+p^2}.\sqrt{3}} = \cos\alpha$, where

$\cos 2\alpha = 2\cos^2\alpha - 1 = \frac{1}{3} \Rightarrow \cos\alpha = \frac{\sqrt{2}}{\sqrt{3}}$, so that $m + n + p = \sqrt{m^2+n^2+p^2}.\sqrt{2}$.

Squaring both sides: $m^2 + n^2 + p^2 + 2mn + 2np + 2pm = 2(m^2 + n^2 + p^2)$

$$\Rightarrow 2mn + 2np + 2pm = m^2 + n^2 + p^2$$

Setting $m = 2n + 2p$ (or equivalent) then gives $2np + (2n + 2p)^2 = (2n + 2p)^2 + n^2 + p^2$

which gives $(n - p)^2 = 0 \Rightarrow p = n,\ m = 4n$.

Thus $\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$, or any non-zero multiple will suffice.

(ii) If you used the above method then you already have this relationship; namely,

$$2uv + 2vw + 2wu = u^2 + v^2 + w^2.$$

Thus, $2xy + 2yz + 2zx = x^2 + y^2 + z^2$ gives all lines inclined at an angle $\cos^{-1}\frac{\sqrt{2}}{\sqrt{3}}$ to OA and hence describes the surface which is a double-cone, vertex at O, having central axis OA.

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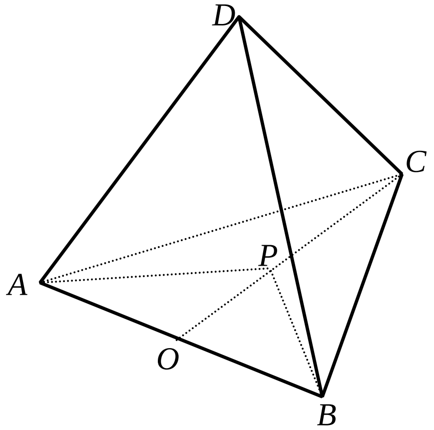
6 Although it seems that 3-dimensional problems are not popular, this is actually a very, very easy question indeed and requires little more than identifying an appropriate right-angled triangle and using some basic trig. and/or *Pythagoras*. There are thus so many ways in which one can approach the three parts to this question that it is difficult to put forward just the one.

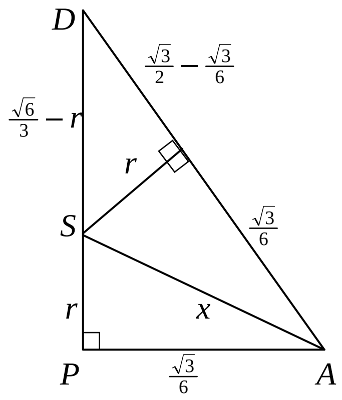
(i) Taking the midpoint of AB as the origin, O, with the x-axis along AB and the y-axis along OC, we have a cartesian coordinate system to help us organise our thoughts.



Then $A = \left(-\frac{1}{2},\ 0,\ 0\right)$, $B = \left(\frac{1}{2},\ 0,\ 0\right)$, $C = \left(0,\ \frac{\sqrt{3}}{2},\ 0\right)$ by trig. or Pythagoras, and $P = \left(0,\ \frac{\sqrt{3}}{6},\ 0\right)$. The standard distance formula then gives PA (or PB) $= \frac{\sqrt{3}}{3}$ and $PD = \frac{\sqrt{6}}{3}$ or $\sqrt{\frac{2}{3}}$.

(ii) The angle between adjacent faces is (e.g.) $\angle DOC = \cos^{-1}\left(\frac{\frac{1}{6}\sqrt{3}}{\frac{1}{2}\sqrt{3}}\right)$ in right-angled triangle DOP, which gives the required answer, $\cos^{-1}\frac{1}{3}$.

(iii)



The centre, S, of the inscribed sphere must, by symmetry, lie on PD, equidistant from each vertex.

By Pythagoras, $x^2 = \frac{1}{12} + \left(\frac{6}{9} - 2\frac{\sqrt{6}}{3}x + x^2\right) \Rightarrow x = \frac{\sqrt{6}}{4}$.
Then $r = x\sin(90^{\circ} - \text{(ii)}) = \frac{1}{3}x = \frac{\sqrt{6}}{12}$.

Alternatively, if you know that the sphere's centre is at The centre of mass of the tetrahedron, the point (S) with position vector $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})$, then the answer is just $\frac{1}{4}DP = \frac{\sqrt{6}}{12}$.

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7 The first two parts of the question begin, helpfully, by saying exactly what to consider in order to proceed, and the material should certainly appear to be routine enough to make these parts very accessible. Where things are going in (iii) may not immediately be obvious but, presumably, there is a purpose to (i) and (ii) which should become clear in (iii).

(i) $y = x^3 - 3qx - q(1+q) \Rightarrow \frac{dy}{dx} = 3(x^2 - q) = 0$ for $x = \pm\sqrt{q}$.

When $x = +\sqrt{q}$, $y = -q\left(\sqrt{q}+1\right)^2 < 0$ since $q > 0$

When $x = -\sqrt{q}$, $y = -q\left(\sqrt{q}-1\right)^2 < 0$ since $q > 0$ **and** $q \neq 1$

Since both TPs below x-axis, the curve crosses the x-axis once only (possibly with sketch)

(ii) $x = u + \frac{q}{u} \Rightarrow x^3 = u^3 + 3uq + 3\frac{q^2}{u} + \frac{q^3}{u^3}$

$$0 = x^3 - 3qx - q(1+q) = u^3 + 3uq + 3\frac{q^2}{u} + \frac{q^3}{u^3} - 3qu - 3\frac{q^2}{u} - q - q^2$$

$$\Rightarrow u^3 + \frac{q^3}{u^3} - q(1+q) = 0 \quad \text{or} \quad \left(u^3\right)^2 - q(1+q)\left(u^3\right) + q^3 = 0$$

$$u^3 = \frac{q(1+q) \pm \sqrt{q^2(1+q)^2 - 4q^3}}{2} = \frac{q}{2}\left\{1 + q \pm \sqrt{1 + 2q + q^2 - 4q}\right\}$$

$$= \frac{q}{2}\left\{1 + q \pm \sqrt{(1-q)^2}\right\} = \frac{q}{2}\left\{1 + q \pm (1-q)\right\} = q \text{ or } q^2$$

giving $u = q^{\frac{1}{3}}$ or $q^{\frac{2}{3}}$ and $x = q^{\frac{1}{3}} + q^{\frac{2}{3}}$

(iii) $\alpha + \beta = p$, $\alpha\beta = q \Rightarrow \alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = p^3 - 3qp$.

One root is the square of the other $\Leftrightarrow \alpha = \beta^2$ or $\beta = \alpha^2 \Leftrightarrow 0 = \left(\alpha^2 - \beta\right)\left(\alpha - \beta^2\right)$.
Then $0 = \left(\alpha^2 - \beta\right)\left(\alpha - \beta^2\right) = \alpha^3 + \beta^3 - \alpha\beta - (\alpha\beta)^2 = p^3 - 3qp - q(1+q)$
$\Leftrightarrow p = q^{\frac{1}{3}} + q^{\frac{2}{3}}$.

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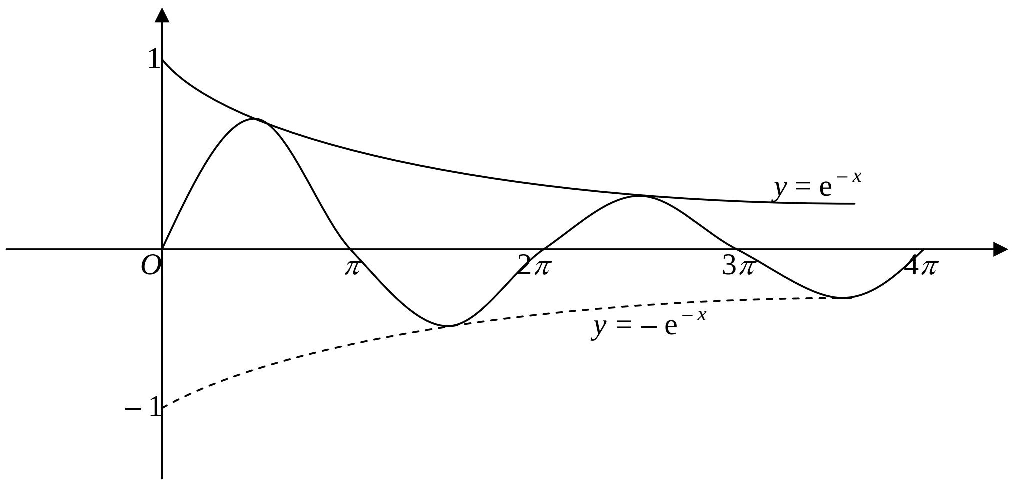
8 When asked to draw sketches of graphs, it is important to note the key features. The first curve is a standard "exponential decay" curve; the second has the extra factor of sin$x$. Now sin$x$ oscillates between –1 and 1, and introduces zeroes at intervals of π. Thus, C_2 oscillates between C_1 and $-C_1$, with zeroes every π units along the x-axis. The x_i that are then introduced are the x-coordinates of C_2 when sin$x$ = 1, and it is important to note that they do not coincide with the maxima of C_2, [It is important to be clear in your description of x_n and x_{n+1} in terms of n as these are going to be substituted as limits into the area integrals that follow.] The integration required to find one representative area will involve the use of "parts", and the final summation looks like it must be that of an infinite GP.



The curves meet each time $\sin x = 1$ when $x = 2n\pi + \frac{\pi}{2}$ (n = 0, 1, 2, …).

Thus $x_n = \dfrac{(4n-3)\pi}{2}$ and $x_{n+1} = \dfrac{(4n+1)\pi}{2}$.

$\int \left(e^{-x}\sin x\right)dx$ attempted by parts $= -e^{-x}.\cos x - \int\left(e^{-x}.\cos x\right)dx$ or $-e^{-x}.\sin x - \int\left(e^{-x}.\sin x\right)dx$

(depending on your choice of '1st' and '2nd' part) $= -e^{-x}.\cos x - \left\{e^{-x}.\sin x + \int\left(e^{-x}.\sin x\right)dx\right\}$.

Then $I = -e^{-x}(\cos x + \sin x) - I$ (by "looping") $= -\frac{1}{2}e^{-x}(\cos x + \sin x)$

$$A_n = \int_{x_n}^{x_{n+1}}\left(e^{-x} - e^{-x}\sin x\right)dx = \left[-e^{-x} + \tfrac{1}{2}e^{-x}(\cos x + \sin x)\right]_{x_n}^{x_{n+1}} \text{ or } \left[\tfrac{1}{2}e^{-x}(\cos x + \sin x - 2)\right]_{x_n}^{x_{n+1}}$$

$$= \tfrac{1}{2}e^{-\frac{1}{2}\pi(4n+1)}(0+1-2) - \tfrac{1}{2}e^{-\frac{1}{2}\pi(4n-3)}(0+1-2) = \tfrac{1}{2}e^{-\frac{1}{2}\pi(4n+1)}\left(-1+e^{2\pi}\right)$$

Note that $A_1 = \frac{1}{2}e^{-\frac{5}{2}\pi}\left(e^{2\pi}-1\right)$ and $A_{n+1} = e^{-2\pi}A_n$ so that $\sum_{n=1}^{\infty}A_n = A_1\left\{1 + \left(e^{-2\pi}\right) + \left(e^{-2\pi}\right)^2 + \ldots\right\}$

$= \frac{1}{2}e^{-\frac{5}{2}\pi}\left(e^{2\pi}-1\right)\times\dfrac{1}{1-e^{-2\pi}} = \frac{1}{2}e^{-\frac{5}{2}\pi}\left(e^{2\pi}-1\right)\times\dfrac{e^{2\pi}}{e^{2\pi}-1}$ (using the $S\infty$ of a GP formula) $= \frac{1}{2}e^{-\frac{1}{2}\pi}$

9 Once you have written down all relevant possible equations of motion, this question is really quite simple; the two results you are asked to prove arise from considering either times or distances to the point of collision. There is, however, one crucial realisation to make in the process, without which further progress is almost impossible; once noted, it seems terribly obvious, yet it probably doesn't usually fall within the remit of standard A-level examination questions.

For P_1, $\ddot{x}_1 = 0$, $\dot{x}_1 = u\cos\alpha$, $x_1 = ut\cos\alpha$, $\ddot{y}_1 = -g$, $\dot{y}_1 = u\sin\alpha - gt$, $y_1 = ut\sin\alpha - \tfrac{1}{2}gt^2$

For P_2, $\ddot{x}_2 = 0$, $\dot{x}_2 = v\cos\beta$, $x_2 = vt\cos\beta$, $\ddot{y}_2 = -g$, $\dot{y}_2 = v\sin\beta - gt$, $y_2 = vt\sin\beta - \tfrac{1}{2}gt^2$

Now P_1 is at its greatest height when $\dot{y}_2 = 0 \Rightarrow t = \dfrac{u\sin\alpha}{g} \Rightarrow y_1 = h = \dfrac{u^2\sin^2\alpha}{2g}$ and it follows that $u\sin\alpha = \sqrt{2gh}$

Note that if the two particles are at the same height at any two distinct times (one of which is $t = 0$ here), then their vertical speeds are the same throughout their motions. Thus $u\sin\alpha = v\sin\beta$.

$y_2 = 0,\ t \neq 0 \Rightarrow t = \dfrac{2v\sin\beta}{g}$. This is the time when P_2 would land. Also, the collision occurs when $x_2 = b \Rightarrow t = \dfrac{b}{v\cos\beta}$ is the time of the collision.

Then $t\left(P_2\ \tfrac{1}{2}\text{–range}\right) < t(\text{collision}) < t\left(P_2\ \text{range}\right)$ **(**or by distances)

$\Rightarrow \dfrac{v\sin\beta}{g} < \dfrac{b}{v\cos\beta} < \dfrac{2v\sin\beta}{g} \Rightarrow \dfrac{v^2\sin\beta\cos\beta}{g} < b < \dfrac{2v^2\sin\beta\cos\beta}{g}$

$\Rightarrow \dfrac{(v\sin\beta)^2}{g}\cot\beta < b < \dfrac{2(v\sin\beta)^2}{g}\cot\beta$. Using $u\sin\alpha = v\sin\beta = \sqrt{2gh}$ then gives

$\dfrac{2gh}{g}\cot\beta < b < \dfrac{4gh}{g}\cot\beta \Rightarrow 2h\cot\beta < b < 4h\cot\beta$.

One could repeat all this work for P_1, but this is not necessary. Since the particles are at their maximum heights simultaneously (see the above reasoning) *and* would achieve their "ranges" simultaneously also, we have $2h\cot\alpha < a < 4h\cot\alpha$.

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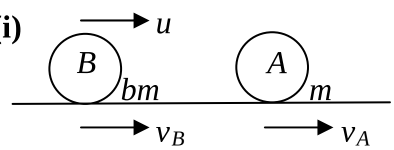
10 I always feel that collisions questions are very simple, since (as a rule) there are only the two main principles – *Conservation of Linear Momentum* and *Newton's Experimental Law of Restitution* – to be applied. Such is the case here. Part (ii) is only rendered more difficult by the introduction of a number of repetitions, and then the question concludes with some pure mathematical work using logarithms.

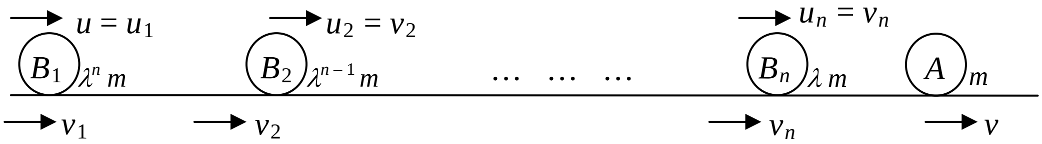
(i)



Using CLM: $bmu = bmv_B + mv_A$.
Using NEL: $u = v_A - v_B$.
Solving simultaneously: $v_A = \dfrac{2bu}{b+1}$ and $v_B = \dfrac{(b-1)u}{b+1}$.

Then $v_A = \left(\dfrac{2}{1+\frac{1}{b}}\right)u \to 2u-$ as $b \to \infty$, and $v_A < 2u$ always.

(ii)



Using the results of (i), $v_2 = u_2 = \left(\dfrac{2\lambda}{\lambda+1}\right)u$; $u_3 = \left(\dfrac{2\lambda}{\lambda+1}\right)u_2 = \left(\dfrac{2\lambda}{\lambda+1}\right)^2 u$; … etc. …

all the way down to $u_n = \left(\dfrac{2\lambda}{\lambda+1}\right)u_{n-1} = \left(\dfrac{2\lambda}{\lambda+1}\right)^{n-1} u$ and $v = \left(\dfrac{2\lambda}{\lambda+1}\right)u_n = \left(\dfrac{2\lambda}{\lambda+1}\right)^n u$.

Since $u_n = \dfrac{2\lambda}{\lambda+1} > 1$, as $\lambda > 1$, it follows that v can be made as large as possible.

In the case when $\lambda = 4$, $v = \left(\frac{8}{5}\right)^n u > 20u$ requires $n \log\left(\frac{8}{5}\right) > \log 20 \Rightarrow n > \dfrac{\log 20}{\log\left(\frac{8}{5}\right)}$.

Now $\log 2 = 0.30103 \Rightarrow \log 8 = 3\log 2 = 0.90309$
and $\log 5 = \log 10 - \log 2 = 1 - 0.30103 = 0.69897$
so that $\log\left(\frac{8}{5}\right) = \log 8 - \log 5 = 0.20412$.

Also $\log 20 = \log 10 + \log 2 = 1 + 0.30103 = 1.30103$, so we have $n > \dfrac{1.30103}{0.20412}$.

Since $6 \times 0.20412 = 1.22472$ and $7 \times 0.20412 = 1.42884$, $n_{\min} = 7$.

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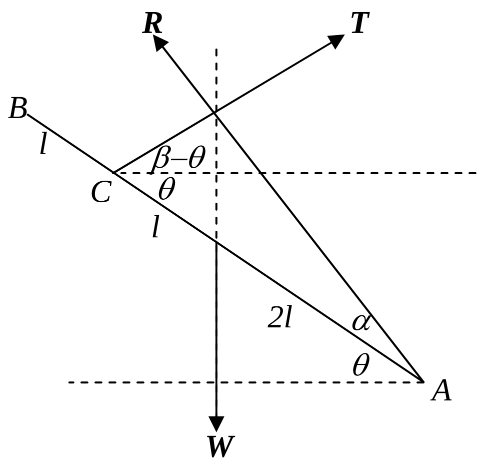
11 A few years ago, a standard "three-force" problem such as this would have elicited responses using *Lami's Theorem*; since this tidy little result seems to have lapsed from the collective A-level consciousness, I shall run with the more popular, alternative *Statics*-question approach of resolving twice and taking moments. In order to get started, however, it is important to have a good, clear diagram suitably marked with correct angles. The later parts of the question consist mostly of trignometric work.



<u>Res.↑</u> $T\sin(\beta-\theta) + R\sin(\alpha+\theta) = W$

<u>Res.→</u> $T\cos(\beta-\theta) = R\cos(\alpha+\theta)$

<u>A↵</u> $W.2l\cos\theta = T.3l\sin\beta$

Substituting to eliminate T's (e.g.) $\Rightarrow T\sin(\beta-\theta) + \dfrac{T\cos(\beta-\theta)}{\cos(\alpha+\theta)}\sin(\alpha+\theta) = \dfrac{3T\sin\beta}{2\cos\theta}$

$$\Rightarrow 2\cos\theta\left(\cos\alpha.\cos\theta - \sin\alpha.\sin\theta\right)\left(\sin\beta.\cos\theta - \cos\beta.\sin\theta\right)$$
$$+ 2\cos\theta\left(\cos\beta.\cos\theta + \sin\beta.\sin\theta\right)\left(\sin\alpha.\cos\theta + \cos\alpha.\sin\theta\right)$$
$$= 3\sin\beta\left(\cos\alpha.\cos\theta - \sin\alpha.\sin\theta\right)$$

Dividing by $\cos\theta \; \cos\alpha \; \cos\beta$

$$\Rightarrow \; 2(\cos\theta - \tan\alpha.\sin\theta)(\tan\beta.\cos\theta - \sin\theta) + 2(\cos\theta + \tan\beta.\sin\theta)(\tan\alpha.\cos\theta + \sin\theta)$$
$$= 3\tan\beta(1 - \tan\alpha.\tan\theta)$$

Multiplying out, cancelling and collecting up terms, and then dividing by $\tan\alpha \; \tan\beta$ then gives the required answer $2\cot\beta + 3\tan\theta = \cot\alpha$.

$$\theta = 30^{\circ},\; \beta = 45^{\circ} \;\Rightarrow\; \cot\alpha = 2.1 + 3.\frac{1}{\sqrt{3}} = 2 + \sqrt{3},$$

and $\tan 15^{\circ} = \tan(60^{\circ} - 45^{\circ}) = \dfrac{\sqrt{3}-1}{1+\sqrt{3}} = \dfrac{\left(\sqrt{3}-1\right)^2}{3-1} = 2 - \sqrt{3} = \dfrac{1}{2+\sqrt{3}}$.

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12 In some ways, the pdf $f(x)$ couldn't be much simpler, consisting of just two horizontal straight-line segments (in the non-zero part). Part (i) is then relatively routine – use "total prob. = 1" to find the value of *k*, before proceeding to find *E*(*X*); and the trickiest aspect of (ii) is in the inequalities work. You also need to realise that the median could fall in either of the two non-zero regions. For (iii), it is necessary only to follow through each possible value of *M* relative to *E*, the expectation.

Since the pdf is only non-zero between 0 & 1, and the area under its graph = 1, if a, b are both < (>) 1 then the total area will be < (>) 1. Since we are given that $a > b$, it must be the case that $a > 1$ and $b < 1$.

(i) $1 = \displaystyle\int_0^1 f(x)\,dx = \int_0^k a\,dx + \int_k^1 b\,dx = \big[ax\big]_0^k + \big[bx\big]_k^1 = ak + b - bk \;\Rightarrow\; k = \frac{1-b}{a-b}$.

$$E(X) = \int_0^1 x f(x)\,dx = \int_0^k ax\,dx + \int_k^1 bx\,dx = \left[\frac{ax^2}{2}\right]_0^k + \left[\frac{bx^2}{2}\right]_k^1 = \frac{ak^2}{2} + \frac{b}{2} - \frac{bk^2}{2}$$
$$= \frac{b}{2} + \frac{(a-b)}{2} \times \left(\frac{1-b}{a-b}\right)^2 = \frac{ba - b^2 + 1 - 2b + b^2}{2(a-b)} = \frac{1-2b+ab}{2(a-b)}.$$

(ii) If $ak \ge \tfrac{1}{2}$ (i.e. $M \in (0, k)$) then $\dfrac{a-ab}{a-b} \ge \tfrac{1}{2} \;\Rightarrow\; 2a - 2ab \ge a - b \;\Rightarrow\; a + b \ge 2ab$

and $aM = \tfrac{1}{2}$ or $M = \dfrac{1}{2a}$.

If $ak \le \tfrac{1}{2}$ (i.e. $M \in (k, 1)$), and noting that this is equivalent to $a + b \le 2ab$,

then $ak + (M - k)b = \tfrac{1}{2}$ or $(1 - M)b = \tfrac{1}{2} \;\Rightarrow\; M = 1 - \dfrac{1}{2b}$

(iii) If $a + b \ge 2ab$, then $\mu - M = \dfrac{1-2b+ab}{2(a-b)} - \dfrac{1}{2a} = \dfrac{a - 2ab + a^2b - a + b}{2a(a-b)} = \dfrac{b(1-a)^2}{2a(a-b)} > 0$

and the required result follows.

If $a + b \le 2ab$, then $\mu - M = \dfrac{1-2b+ab}{2(a-b)} - 1 + \dfrac{1}{2b} = \dfrac{b - 2b^2 + ab^2 - 2ab + 2b^2 + a - b}{2b(a-b)}$

$$= \frac{a(1-b)^2}{2b(a-b)} > 0 \text{ as required.}$$

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13 This question is really little more than examining the various cases that arise for each outcome and then doing a little bit of work algebraically. The result of part (i) is somewhat counter-intuitive, in that Rosalind should choose to play the more difficult opponent twice, while one intutively feels she should be playing the easier opponent. The real issue, however, is that she needs to beat both opponents (and not just win one game): examining the probabilities algebraically makes this very obvious. Part (ii) is a nice adaptation, where there is a cut-off point separating the cases when one strategy is always best from another situation when either strategy 1 or 2 can be best. Here, it is most important to demonstrate that the various conditions hold, and not simply state a couple of probabilities and hope they do the job. [It is perfectly possible to do (iii) by "trial-and-error", but I have attempted to reproduce below an approach which incorporates a method for deciding the matter.]

(i) $P(W_{PPQ}) = P(W_P\ W_Q\ –) + P(L_P\ W_Q\ W_P) = p\ .\ q.\ 1 + (1 – p)qp = pq(2 – p)$.
Similarly, $P(W_{PQQ}) = pq(2 – q)$ and $P(W_{PPQ}) – P(W_{PQQ}) = pq(q – p) > 0$ since $q > p$. Thus, $P(W_{PPQ}) > P(W_{PQQ})$ for all p, q and "Ros plays Pardeep twice" is always her best strategy.

(ii) SI: $P(W_1) = P(W_Q\ W_P\ – –) + P(W_Q\ L_P\ W_P\ –) + P(W_Q\ L_P\ L_P\ W_P)$

$= pq + pq(1 – p) + pq(1 – p)^2$ or $pq(3 – 3p + p^2)$

SIII: $P(W_3) = pq(3 – 3q + q^2)$ similarly.

SII: $P(W_2) = P(W_P\ W_Q\ – –) + P(L_P\ W_P\ W_Q\ –) + P(W_P\ L_Q\ W_Q\ –) + P(L_P\ W_P\ L_Q\ W_Q)$
$= pq + pq(1 – p) + pq(1 – q) + pq(1 – p)(1 – q)$
$= pq(4 – 2p – 2q + pq)$ or $pq(2 – p)(2 – q)$.

$P(W_1) – P(W_3) = pq(q – p)\big(3-[p+q]\big) > 0$ since $q > p$ and $p + q < 2 < 3$ so that **SI** is always better than **S3**

$P(W_1) – P(W_2) = pq\left(p^2 - p - 1 - pq + 2q\right) = pq\big((2-p)(q-p)-(1-p)\big)$

> 0 whenever $q – p > \dfrac{1-p}{2-p} = 1 - \dfrac{1}{2-p}$.

Now $p + \frac{1}{2} < q < 1 \Rightarrow 0 < p < \frac{1}{2} \Rightarrow \frac{1}{3} < 1 - \dfrac{1}{2-p} < \frac{1}{2}$, so that **SI** always better than **SII** when $q – p > \frac{1}{2}$.

$P(W_1) – P(W_2) >< 0 \Leftrightarrow q – p >< \dfrac{1-p}{2-p}$.

Take $p = \frac{1}{4}, q = \frac{1}{2} \Rightarrow q – p = \frac{1}{4} < \frac{1}{2}$ and $\dfrac{1-p}{2-p} = \frac{3}{7} > \frac{1}{4}$ so **SII** is better than **SI**.

Take $p = \frac{1}{4}, q = \frac{3}{4} - \varepsilon \Rightarrow q – p = \frac{1}{2} - \varepsilon < \frac{1}{2}$ and $\dfrac{1-p}{2-p} = \frac{3}{7}$ so choosing

$\varepsilon < \frac{3}{7} - \frac{1}{2} = \frac{1}{14}$ (say $\frac{1}{16}$) will give $p = \frac{1}{4}, q = \frac{11}{16}$ and $q – p = \frac{7}{16} > \dfrac{1-p}{2-p} = \frac{3}{7}$ so that **SI** is better than **SII**.

[I believe that $q – p > k$ has $k = \frac{1}{2}$ as the least positive k which *always* gives **SI** better than **SII**, but it is a long time ago that the problem was originally devised and I may be wrong.]